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Normal forms of real symmetric systems with multiplicity

Dedicated to Prof. Jaap Korevaar on the occasion of his 70th birthday

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ABSTRACT

A normal form is given for real symmetric systems of linear partial differential equations, at points where the principal symbol has a two-dimensional kernel under assumptions which apply to the generic case.

1. INTRODUCTION

This paper takes a step towards understanding the propagation of polarization of solutions of real, symmetric linear systems, $\mathbf{Q}u = 0$, of partial differential equations. We will show that the operator $\mathbf{Q}$ can be brought into the very simple standard form

$$(1.1)\qquad \begin{pmatrix} D_1 + D_2 & x_2 D_3 \\ x_2 D_3 & \pm(D_1 - D_2) \end{pmatrix}, \quad D_j = \frac{1}{i}\frac{\partial}{\partial x_j}.$$

This normal form can be achieved by splitting off elliptic summands, multiplying by invertible pseudodifferential operators and conjugating with invertible Fourier integral operators. The normal form is obtained modulo terms for which the full Taylor expansion of the principal symbol vanishes at every point of Σ. Here Σ is the subset of the cotangent bundle, where the principal symbol has a zero eigenvalue of multiplicity higher than one. The construction is microlocal, that is, in some conic neighborhood of a given point in the cotangent bundle. Of course, one would hope that the standard system is easier to investigate than the system in its original form.

In order to explain the assumptions under which the results can be proved, we recall that the propagation of polarization can be paraphrased mathematically as the behavior of asymptotic high-frequency solutions of the m-th order system $\mathbf{Q}u = 0$. If

$$u(x) = \mathrm{e}^{\mathrm{i}\tau\langle x, \xi\rangle}\, a(x)$$

is a high frequency wave, with frequency τ, phase covector ξ and amplitude vector $a(x)$, then

$$(\mathbf{Q}u)(x) \sim \tau^m\, \mathrm{e}^{\mathrm{i}\tau\langle x, \xi\rangle}\, Q(x, \xi)\, a(x),$$

asymptotically as $\tau \to \infty$. Here $Q(x, \xi)$ is a matrix, which is called the *principal symbol* of the operator $\mathbf{Q}$, which is intrinsically defined on the *cotangent bundle* T^*M of M, the phase space of classical mechanics, on which (x, ξ) are canonical coordinates. One says that $u(x)$ is an *asymptotic solution* if $\mathbf{Q}(u)$ is of order τ^l with $l < m$.

The operator $\mathbf{Q}$ is called *elliptic at* (x, ξ) if $Q(x, \xi)$ is invertible. Clearly, $u(x)$ can only be an asymptotic solution if the amplitude vector $a(x)$ belongs to $\ker Q(x, \xi)$, the *polarization space* of $\mathbf{Q}$ at (x, ξ). Therefore, high-frequency solutions with nonzero amplitude vector can only occur if (x, ξ) lies in

$$N = \{(x, \xi) \in T^*M;\ \det Q(x, \xi) = 0\}$$

the *characteristic set* of $\mathbf{Q}$.

At the points of N where $\det Q$ has simple zeros, the polarization space is one-dimensional and one can reduce the study of the operator to that of the scalar case, cf. Dencker [8]. In turn the scalar case with simple zeros can be reduced to the study of the operator $\partial/\partial t$, using multiplication by elliptic operators and conjugation by invertible Fourier integral operators, see Duistermaat and Hörmander [13, Sec. 6]. A generic scalar operator will only have these, so called, simple characteristics, see Nuij [22].

For effects which are truly specific for systems, we therefore must turn to the subset Σ of the points $(x, \xi) \in T^*M$, at which $\det Q$ has zeros with multiplicity more than one. This location is called the optical or acoustical axis by physicists when they consider the Maxwell equations or the equations for waves in elastic media. It has been known for some time that these multiplicities sometimes occur for topological reasons and are present generically, see Lax [19], John [14] and Hörmander [18].

Rather than investigating the situation for any system we shall assume that for each $(x, \xi) \in T^*M$, the principal symbol $Q(x, \xi)$ is a real and symmetric matrix. This is the case for many systems in mathematical physics, in particular for all systems arising from variational problems, even when arbitrary lower order terms are added as perturbations. Under explicit and generic nondegeneracy conditions, stated in Section 3, we obtain the normal form in a conic neighborhood of $(x, \xi) \in \Sigma$.

The two sign choices in (1.1) lead to drastically different behaviour of the solutions. For the plus sign, the operator is hyperbolic with respect to the variable x_1. Close to Σ, the bicharacteristic curves in the regular part of the characteristic set N form helices, narrowly winding around smooth curves in Σ. Along with it, the polarization space rotates rapidly. For the minus sign, the operator is hyperbolic with respect to x_2. The bicharacteristic curves in N approach Σ and bounce away like a hyperbola approaching the intersection of its asymptotes. During the change of direction, the polarization space makes a quarter turn.

When thinking of systems with multiplicity, one first thinks of the phenomenon of conical refraction of light, in which a thin lightbeam changes into a cone of light upon reaching a bi-axial crystal. This was predicted by Hamilton and experimentally verified by Lloyd in 1837, see Hamilton [9] and Born and Wolf [5, p. xxii]. For studies with modern analytical tools see for instance Melrose and Uhlmann [21] and Dencker [7]. The normal forms for 2 by 2 systems with conical refraction are

$$\begin{pmatrix} D_1 + D_2 & D_3 \\ D_3 & D_1 - D_2 \end{pmatrix}.$$

The non-generic aspect of this system lies in the fact that it is independent of the coordinates in the base making the singular part of the characteristic variety involutive.

Our paper is organised as follows. In Section 2 we discuss how splitting off elliptic factors leads to 2×2 systems. In Section 3 we study some basic symplectic geometry of the symbol in order to formulate the assumptions under which the normal form can be achieved. In Section 4 we state the result and begin the proof which we finish in Section 5. We finish, in Section 6, by checking that Maxwell's equations and the equations for elastodynamics satisfy the genericity assumptions and realise both signs.

This work was started subsequent to listening to a lecture delivered by V.I. Arnol'd in Utrecht in the spring of 1990. We thank him for his inspiration.

2. SPLITTING OFF ELLIPTIC SUMMANDS

Let $E \to M$ be a k-dimensional smooth real vector bundle over an n-dimensional paracompact smooth manifold M. We will study a linear pseudodifferential operator $\mathbf{Q}$ of order m, acting on the space $\Gamma(M, E)$ of smooth sections over M. Since our constructions are (micro-)local, we use a local trivialization of E, in which $\mathbf{Q}$ can be identified with a $k \times k$-matrix of pseudodifferential operators. The principal symbol $Q(x, \xi)$ of $\mathbf{Q}$ is a $k \times k$-matrix, depending smoothly on $(x, \xi) \in T^*M \setminus 0$. It is homogeneous of degree m in the sense that

$$Q(x, \tau\xi) = \tau^m Q(x, \xi), \quad \tau > 0.$$

The half-line $\{(x, \tau\xi) \mid \tau > 0\}$ is called the *cone axis* through (x, ξ), and subsets of T^*M are called conic when they are the union of cone axes. For example, the characteristic set N is a closed conic subset of $T^*M \setminus 0$, because $\det Q$ is homogeneous of degree km.

Near elliptic points, $\mathbf{Q}$ has a pseudodifferential parametrix $\mathbf{R} = \mathbf{Q}^{-1}$ of order $-m$, in the sense that $\mathbf{QR} \sim \mathbf{RQ} \sim \mathbf{I}$ near (x, ξ). Here the expression '$\mathbf{A} \sim \mathbf{B}$ near (x, ξ)', for pseudodifferential operators $\mathbf{A}, \mathbf{B}$, means that $\mathbf{A} - \mathbf{B}$ is smoothing (has order $-\infty$) in a conic neighborhood of (x, ξ). We start with a basic, simple observation (compare Dencker [7, proof of Prop. 2.5] and Hörmander [18, Lemma 2.1]).

Lemma 1. *Suppose that we have the block decomposition*

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{ll} & \mathbf{Q}_{lr} \\ \mathbf{Q}_{rl} & \mathbf{Q}_{rr} \end{pmatrix}. \tag{2.1}$$

in which $\mathbf{Q}_{ij}$ has i rows and j columns, $r = k - l$, and $\mathbf{Q}_{rr}$ is elliptic in a conic open subset U of $T^*M \setminus 0$. Then there exist elliptic pseudodifferential operators $\mathbf{A}, \mathbf{B}$ of order 0 *such that in U*

$$\mathbf{AQB} \sim \begin{pmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{Q}_{rr} \end{pmatrix}.$$

Here $\mathbf{P}^* - \mathbf{P}$ is of order $\leq \mu$ if $\mathbf{Q}^* - \mathbf{Q}$ is of order $\leq \mu$, the principal symbol $P(x, \xi)$ of $\mathbf{P}$ is real if the principal symbol $Q(x, \xi)$ of $\mathbf{Q}$ is real, and finally $P(x, \xi) = 0$ if $\dim \ker Q(x, \xi) = l$.

Proof. Take

$$\mathbf{A} \sim \begin{pmatrix} \mathbf{I} & -\mathbf{Q}_{lr}\mathbf{Q}_{rr}^{-1} \\ 0 & \mathbf{I} \end{pmatrix}, \qquad \mathbf{B} \sim \begin{pmatrix} \mathbf{I} & 0 \\ -\mathbf{Q}_{rr}^{-1}\mathbf{Q}_{rl} & \mathbf{I} \end{pmatrix}.$$

Then we get the desired form for $\mathbf{AQB}$, with

$$\mathbf{P} \sim \mathbf{Q}_{ll} - \mathbf{Q}_{lr}\mathbf{Q}_{rr}^{-1}\mathbf{Q}_{rl}. \qquad \square$$

In the situation of Lemma 1 we will say that the $l \times l$ operator $\mathbf{P}$ is *split off* from the $k \times k$ operator $\mathbf{Q}$, and refer to the process as splitting off an elliptic summand after multiplication with elliptic factors. This process is compatible with the investigation of propagation of singularities. To see this, note that

$$u := \mathbf{B}\begin{pmatrix} v \\ w \end{pmatrix}$$

is smooth if and only if v and w are. Then $\mathbf{Q}u$ is smooth if and only if $\mathbf{P}v$ and w are smooth. In the context of asymptotic high-frequency solutions, 'smooth' has to be replaced by 'asymptotically small'. Note also that the principal symbol $B(x, \xi)$ of $\mathbf{B}$ is an isomorphism from $\ker P(x, \xi) \oplus 0$ to the polarization space $\ker Q(x, \xi)$ of $\mathbf{Q}$. In particular these spaces have the same dimension.

We have

$$\det Q(x, \xi) = \det P(x, \xi) \det Q_{rr}(x, \xi),$$

in which $\det Q_{rr}(x, \xi)$ is pointwise non-zero. This implies that $\det Q$ and $\det P$ have the same order of zeros. If $\dim \ker Q(x, \xi) = l$, then $P(x, \xi) = 0$, so $\det P$ has a zero of order at least l at (x, ξ). Because the same holds for $\det Q$, we have proved:

Lemma 2. *If* $\dim\ker Q(x,\xi) = l$ *and* $Q(x,\xi)$ *is invertible modulo its kernel, then* $\det Q$ *has a zero of order at least* l *at* (x,ξ).

3. THE GENERICITY ASSUMPTIONS

In the sequel, we will concentrate on a conic neighborhood of a point in the set

$$\Sigma := \{(x,\xi) \in N \mid \dim\ker Q(x,\xi) > 1\}.$$

Note that Σ is a closed, conic subset of N. In this section, we will investigate the geometry of the symbol around Σ, with the purpose of establishing the genericity assumptions under which the normal form can be obtained. We begin with the consequences of Lemma 1 for operators with real symmetric principal symbols.

Corollary 3. *Assume that, possibly after multiplying* $\mathbf{Q}$ *with elliptic factors, the principal symbol* $Q(x,\xi)$ *is a real symmetric* $k \times k$*-matrix, and that*

$$\dim\ker Q(x^0,\xi^0) = 2.$$

Then one can split off a 2×2 *operator* $\mathbf{P}$, *with real symmetric principal symbol* $P(x,\xi)$. *Near* (x^0,ξ^0), *we have*

$$(x,\xi) \in \Sigma \;\Leftrightarrow\; P(x,\xi) = 0 \;\Leftrightarrow\; \dim\ker Q(x,\xi) = 2,$$

and $\det Q$ *has a zero of multiplicity more than one at each point of* Σ.

Moreover, the rank of the Hessian of $\det Q$ at (x^0,ξ^0) *is at most equal to three. If it is equal to three, then there is a conic open neighborhood* U *of* (x^0,ξ^0) *in* $T^*M \setminus 0$, *such that* $\Sigma \cap U$ *is a smooth submanifold of codimension three in* U.

Proof. The symmetry implies that the range R of $Q := Q(x^0,\xi^0)$ is equal to the orthogonal complement of $K = \ker Q$. and $Q\,|_R$ is invertible from R to itself. So, with an orthonormal basis for which K is spanned by the first two basis vectors and R by the remaining $k-2$, we get the situation of Lemma 1 with $l = 2$ and $Q(x,\xi)$ symmetric. This proves the first part of the Corollary.

For the last statement, we observe that we can write the symmetric matrix $P(x,\xi)$ in the form

$$P = P(x,\xi) = \begin{pmatrix} q+r & s \\ s & q-r \end{pmatrix}, \tag{3.1}$$

for uniquely determined smooth functions q, r, s of (x,ξ), homogeneous of degree m in ξ. The set Σ is equal to the intersection of the sets where q, r and s vanish. The scalar symbol equals

$$p(x,\xi) := \det P(x,\xi) = q^2 - r^2 - s^2. \tag{3.2}$$

from which we see that the Hessian of p at each point of Σ has rank at most three, with equality if and only if dq, dr and ds are linearly independent. In turn, this implies that Σ is a smooth codimension 3 submanifold of $T^*M \setminus 0$. The last statement follows because the Hessian of $\det Q$ is a nonzero multiple of the Hessian of p at each point of Σ. $\square$

Our next ingredient is the canonical symplectic form

$$\sigma := \sum_{j=1}^{n} d\xi_j \wedge dx_j$$

in T^*M. Each smooth function f in T^*M defines a Hamiltonian vector field

$$H_f = \sum_{j=1}^{n} \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \xi_j},$$

or, equivalently, in a coordinate independent definition:

$$i_{H_f}\,\sigma = -df.$$

Because σ is closed, it follows from the homotopy formula that the Lie derivative of σ with respect to any Hamiltonian vector field is equal to zero or, equivalently, that σ is invariant under the flow of H_f.

If g is another smooth function, then we shall also use the *Poisson brackets*

$$\{f, g\} := \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j} - \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} = -H_f\, g,$$

which define a Lie algebra structure on $C^\infty(M)$.

We now turn to the Hamiltonian vector field of p, the solution curves of which on $N \setminus \Sigma$ are called the *bicharacteristic curves.* From (3.2) we see that

$$H_p = 2(qH_q - rH_r - sH_s),$$

vanishes at $(x, \xi) \in \Sigma$, so it has an invariantly defined linearization

$$L = 2(dq \otimes H_q - dr \otimes H_r - ds \otimes H_s)$$

at (x, ξ), which is a linear endomorphism of the tangent space $\mathcal{T} := T_{(x,\xi)}(T^*M)$. Alternatively L can be regarded as the Hamiltonian vector field on $\mathcal{T}$ of the quadratic part of the Taylor expansion of p at (x, ξ). The rank of L is equal to the rank of the Hessian of p at (x, ξ). It is at most equal to three, with equality if and only if H_q, H_r and H_s are linearly independent at (x, ξ). We will assume this in the sequel.

Notice that the image $R := \operatorname{im} L$ is spanned by H_q, H_r and H_s at (x, ξ). On the other hand, $K := \ker L$ is equal to the intersection of the kernels of dq, dr and ds at (x, ξ), so equal to the tangent space to Σ at (x, ξ). Now the fact that the flow of H_p leaves σ invariant implies that L is an infinitesimally symplectic transformation, that is,

$$\sigma(Lu, v) + \sigma(u, Lv) = 0, \quad u, v \in \mathcal{T}.$$

It follows that the range of L is equal to the symplectic orthogonal complement of the kernel of L. Now Σ is said to be *involutive* at (x, ξ) if the symplectic orthogonal complement of $T_{(x,\xi)}\Sigma$ is contained in $T_{(x,\xi)}\Sigma$. In our situation this would mean that $R \subset K$, or $L^2 = 0$, so that the assumption that L is not nilpotent implies that Σ is not involutive at (x, ξ).

If we restrict σ to R, then its kernel is equal to $R \cap K$ and σ induces a non-degenerate antisymmetric bilinear form on $R/(R \cap K)$. Its dimension is even and at most three, so we either have $R = R \cap K$, corresponding to the case that Σ is involutive at (x, ξ), or $\dim R/(R \cap K) = 2$, in which case σ is a nonzero 2-form on $R/(R \cap K)$. The flow of L preserves the form, hence L is traceless on $R/(R \cap K)$. Therefore, either L has two purely imaginary opposite eigenvalues or two real opposite eigenvalues.

Note that $\dim(R \cap K) = 1$, so there is always a zero eigenvalue for $L|_R$, which actually leads to a nilpotent part of L in T. For instance in the case that L is an infinitesimal rotation, this causes the bicharacteristic curves near Σ to be narrowly winding helices along the curves which are tangent to $R \cap K$.

On the basis in R of the vectors H_q, H_r and H_s at (x, ξ), the matrix of L is equal to

$$\begin{pmatrix} 0 & -2\{r,q\} & -2\{s,q\} \\ 2\{q,r\} & 0 & 2\{s,r\} \\ 2\{q,s\} & 2\{r,s\} & 0 \end{pmatrix}. \tag{3.3}$$

By computing the characteristic polynomial one recovers that it has a zero eigenvalue and that the two other eigenvalues λ satisfy

$$\lambda^2 = 4(\{r,q\}^2 + \{s,q\}^2 - \{r,x\}^2). \tag{3.4}$$

Hence, the condition that L is not nilpotent is equivalent to the condition that the right hand side of (3.4) is nonzero. If it is negative, then L defines an infinitesimal rotation in $R/(R \cap K)$, whereas L induces a hyperbolic area preserving flow in $R/(R \cap K)$ if the right hand side of (3.4) is positive.

Because $\det Q$ is a nonzero multiple of $\det P$, we get that the linearization $L_Q(x, \xi)$ of the Hamiltonian flow of $\det Q$ at $(x, \xi) \in \Sigma$ is a nonzero multiple of L, so all the conditions can be formulated in terms of Q, and are invariant under multiplication by elliptic factors and splitting off elliptic summands. We are now ready to formulate the assumptions under which we will derive our normal form. These coincide with the assumptions made by Ivrii [11], [12, Thm. 3.1 and 3.4], in his investigation of propagation of singularities and by Arnol'd [3, Sec. 8.1–8.4], [1], [2] in his study of the normal form of the characteristic set N near Σ.

Assumption 4. (1) **Q** *has a real and symmetric principal symbol* $Q(x, \xi)$.

(2) $\dim\ker Q(x^0, \xi^0) = 2$. *This implies that* $\det Q$ *has a zero of multiplicity at least two at* (x^0, ξ^0), *which makes that the linearization* $L = L_Q$ *of the Hamiltonian vector field of* $\det Q$ *at* (x^0, ξ^0) *is invariantly defined.*

(3) *The rank of* L *is equal to three and* L *is not nilpotent.*

(4) *The direction of the cone axis, given by the Euler vector field*

$$E = \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial \xi_j},$$

is not contained in the range of L.

(5) $n = \dim M \geq 3$.

It is clear from the above description, that the assumptions are of a generic nature. More precisely a generic, real symmetric pseudo differential principal matrix symbol will meet the variety of symmetric matrices with corank 2 transversely. The same transversality holds for differential operators, cf. Arnol'd [3], Khesin [15].

4. FORMAL NORMAL FORMS

Assumption 4, combined with Lemma 1, enables one to split off an elliptic summand and reduce to a two system with principal symbol as in (3.1). In this section we will determine the normal forms for these 2×2 systems. We will say that a smooth function R is *flat at* Σ if the full Taylor expansion of R vanishes at each point of Σ. The main theorem of the paper is

Theorem 5. *Let* $\mathbf{P}$ *be* 2×2 *system with a symmetric symbol which satisfies Assumption* 4 *at a point* $(x^0, \xi^0) \in T * M$ *of multiplicity two. There is a smooth canonical transformation* f*, homogeneous of degree one with respect to the* ξ*-variables, from a conic open neighborhood* V *of* $(0, dx_3) \in T^*\mathbb{R}^n$ *to a conic open neighborhood* U *of* (x^0, ξ^0)*, and a smooth mapping*

$$A : U \to GL(2, \mathbb{R}),$$

homogeneous of degree $(1-m)/2$*, such that*

$$(APA^t)(f(x,\xi)) = \begin{pmatrix} \xi_1 + \xi_2 & x_2\xi_3 \\ x_2\xi_3 & \pm(\xi_1 - \xi_2) \end{pmatrix} + R(x,\xi), \tag{4.1}$$

where the remainder term R *is flat at*

$$f^{-1}(\Sigma) = \{(x,\xi) \in V \,|\, \xi_1 = \xi_2 = x_2 = 0\}.$$

The plus sign in (4.1) *corresponds to the case that the nonzero eigenvalues* λ *of* L*, cf. Assumption* 4 *and* (3.4)*, are purely imaginary, while the minus sign occurs if these eigenvalues are real.*

The main step in achieving the normal form is:

Proposition 6. *There is a* $GL(2,\mathbb{R})$*-valued function* A *defined in a conical neighborhood* U *of* $(x_0, \xi_0) \in \Sigma$*, homogeneous of degree* $1/4 - m/2$*, such that* APA^t *equals*

$$\begin{pmatrix} \tilde{q} + \tilde{r} & \tilde{s} \\ \tilde{s} & \tilde{q} - \tilde{r} \end{pmatrix}$$

with:

$$\begin{array}{llll} \{\tilde{q},\tilde{r}\} = \{\tilde{q},\tilde{s}\} = 0 & \{\tilde{s},\tilde{r}\} = \pm 1 & \textit{in the + case} \\ \{\tilde{q},\tilde{r}\} = \{\tilde{s},\tilde{r}\} = 0 & \{\tilde{s},\tilde{q}\} = 1 & \textit{in the} - \textit{case} \end{array} \tag{4.2}$$

where the equalities hold modulo functions which are flat at $U \cap \Sigma$*.*

Proof. The proof is the subject of Section 5. □

Theorem 5 is now proved from Propostion 6 in the following way. In the $++$ case, write

$$\xi_1 = \lambda \tilde{q}, \quad \xi_2 = \lambda \tilde{r}, \quad x_2 = \tilde{s}/\lambda, \quad \xi_3 = \lambda^2,$$

in which λ is as in Proposition 7 below. By the homogeneous Darboux theorem, see e.g. Hörmander [17, Thm. 21.1.9], we can locally extend these coordinates to a set of coordinates which define the desired homogeneous canonical transformation. The factor $1/\lambda$ in front of the whole matrix is eliminated if we replace A with $\lambda^{1/2}A$.

In the $+-$ case, we switch back to the previous case by means of the homogeneous canonical transformation

$$\xi_2 \mapsto x_2\xi_3, \quad x_2 \mapsto -\xi_2/\xi_3, \quad \xi_3 \mapsto -\xi_3, \quad x_3 \mapsto -x_3 - x_2\xi_2/\xi_3.$$

Finally, in the $-$ case, we use

$$\xi_1 = \lambda \tilde{r}, \quad \xi_2 = \lambda \tilde{q}, \quad x_2 = \tilde{s}/\lambda, \quad \xi_3 = \lambda^2.$$

Note that in general the canonical transformation does not respect the fibration

$$(x, \xi) \mapsto x : T^*\mathbb{R}^n \to \mathbb{R}^n.$$

Proposition 7. *Let $\tilde{q}$, $\tilde{r}$, $\tilde{s}$ satisfy* (4.2), *and assume they are homogeneous of degree $\frac{1}{2}$. There is a positive smooth function λ, homogeneous of degree $\frac{1}{2}$, such that $\{\lambda, \tilde{q}\}$, $\{\lambda, \tilde{r}\}$, and $\{\lambda, \tilde{s}\}$ are flat at Σ.*

Proof. (Compare Roels & Weinstein [23].) The Hamiltonian vectorfields $H_{\tilde{q}}$, $H_{\tilde{r}}$ and $H_{\tilde{s}}$ commute modulo flat terms. This is based on the Jacobi identity for Poisson brackets, which is equivalent to

$$[H_f, H_g] = H_{\{g,f\}}.$$

By adding flat terms to $\tilde{q}$, $\tilde{r}$ and $\tilde{s}$ we can find $\hat{q}$, $\hat{r}$ and $\hat{s}$, such that $H_{\hat{q}}$, $H_{\hat{r}}$ and $H_{\hat{s}}$ commute. For instance, in the $+$ case, we may take $\hat{q} = \tilde{q}$ and take $\hat{r}$ equal to the solution of $\{\hat{q}, \hat{r}\} = 0$, with $\hat{r} = \tilde{r}$ on a conic codimension one submanifold, which is transversal to $H_{\hat{q}}$. Because $H_{\hat{q}}$ is tangent to Σ, $\hat{r} - \tilde{r}$ is flat at Σ. Next let S be a conic codimension two submanifold of T^*M, transversal to $H_{\hat{q}}$ and $H_{\hat{r}}$, such that $S \cap \Sigma$ is a codimension one submanifold of Σ. The desired $\hat{s}$ is obtained by taking $\{\hat{q}, \hat{s}\} = 0$, $\{\hat{s}, \hat{r}\} = \pm 1$ and $\hat{s} = \tilde{s}$ on Σ.

The Euler vectorfield E in Assumption 4 satisfies for any functions f, g:

$$E\{f, g\} = \{Ef, g\} + \{f, Eg\} - \{f, g\}. \tag{4.3}$$

Let M_0 be a codimension 4 submanifold of T^*M through $(x^0, \xi^0) \in \Sigma$, transverse to $H_{\hat{q}}$, $H_{\hat{r}}$, $H_{\hat{s}}$, E. We choose λ equal to one on M_0 and extend it to a function homogeneous of degree $\frac{1}{2}$ on $M_1 = \mathbb{R}_{>0} \cdot M_0$, using the multiplicative action $(x, \xi) \mapsto (x, \tau\xi)$, $\tau > 0$. The requirement that λ Poisson-commutes with $\hat{q}, \hat{r}, \hat{s}$ defines it on an open subset of T^*M. From the above relation one deduces that for $f = \hat{q}, \hat{r}, \hat{s}$:

$$\{f, E\lambda - \tfrac{1}{2}\lambda\} = 0$$

using the homogeneity of f. Thus λ is homogeneous of degree $\frac{1}{2}$ everywhere. $\square$

The formal normal forms will be achieved by induction on the order. The first proposition will establish the vanishing of the relevant Poisson bracket up to first order. Before proceeding with the induction step we will introduce the usual machinery associated with filtrations by orders.

In this section we have use for the function

$$e = q\{r, s\} + r\{s, q\} + s\{q, r\}.$$

At every point of Σ the Hamilton vectorfield of e spans the nullspace of the restriction to Σ of the symplectic form. This subspace is equal to the intersection of the kernel and range of L, c.f. (3.3). Recall that we wrote our symmetric symbol matrix as

$$P = \begin{pmatrix} q+r & s \\ s & q-r \end{pmatrix}.$$

With $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ an invertible matrix-valued function homogeneous of degree 0, we can write

$$APA^t = \begin{pmatrix} \tilde{q}+\tilde{r} & \tilde{s} \\ \tilde{s} & \tilde{q}-\tilde{r} \end{pmatrix}$$

where

$$\begin{aligned} \tilde{q} &= \tfrac{1}{2}(a^2+b^2+c^2+d^2)\,q + \tfrac{1}{2}(a^2-b^2+c^2-d^2)\,r + (ab+cd)\,s \\ \tilde{r} &= \tfrac{1}{2}(a^2+b^2-c^2-d^2)\,q + \tfrac{1}{2}(a^2-b^2-c^2+d^2)\,r + (ab-cd)\,s \\ \tilde{s} &= (ac+bd)\,q + (ac-bd)\,r + (ad+bc)\,s \end{aligned} \tag{5.1}$$

Lemma 8. *We can find A and a conical neighborhood of (x, ξ) such that on $\Sigma \cap U$*

$$\begin{aligned} &\{\tilde{q}, \tilde{r}\} = \{\tilde{q}, \tilde{s}\} = 0 \quad \{\tilde{s}, \tilde{r}\} = \pm 1 \quad \textit{in the + cases} \\ &\{\tilde{q}, \tilde{r}\} = \{\tilde{s}, \tilde{r}\} = 0 \quad \{\tilde{s}, \tilde{q}\} = 1 \quad\;\; \textit{in the − case.} \end{aligned} \tag{5.2}$$

Proof. We first deal with the +-case and then indicate what modifications one makes for the minus case. The equations $\{\tilde{q}, \tilde{r}\} = \{\tilde{q}, \tilde{s}\} = 0$ on Σ will be satisfied if $\tilde{q}$ is a constant multiple of the function e. This is the case if

$$\begin{aligned} &\tfrac{1}{2}(a^2+b^2+c^2+d^2) = \mu\{r, s\} \\ &\tfrac{1}{2}(a^2-b^2+c^2-d^2) = \mu\{s, q\} \\ &(ab+cd) = \mu\{q, r\} \end{aligned} \tag{5.3}$$

for a constant μ. To see that this equation can be solved observe that the map:

$$A \to A^t A = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix}$$

exhibits a locally trivial fibration $GL(2, \mathbb{R}) \to O(2) \setminus GL(2, \mathbb{R})$. Its image consists of the symmetric matrices with 2 positive eigenvalues, a simply connected component of $\mathbb{R}^3$ minus a cone. A choice of sign for μ ensures that a solution can be found.

Assuming that this has been done, i.e. that $\{q,r\} = \{q,s\} = 0$ on Σ, let $A = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ to get:

$$\tilde{q} = a^2 q \qquad \tilde{r} = a^2 r \qquad \tilde{s} = a^2 s.$$

This gives $\{\tilde{q},\tilde{r}\} = \{\tilde{q},\tilde{s}\} = 0$ and $\{\tilde{r},\tilde{s}\} = a^4\{r,s\}$ on Σ we have that (5.3) is fully met.

The $-$-case is done similarly, swapping the role of q and r by using the fibration:

$$A \to A^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A.$$

Here we get all invertible symmetric matrices with negative determinant and we can solve an equation similar to the one above without choosing a sign. $\square$

We will use the following spaces of formal functions. Observe that q, r, s are coordinates transverse to Σ. Define the ring:

$$\mathcal{R} = C^\infty(\Sigma) \otimes_{\mathbb{R}} \mathbb{R}[[q,r,s]],$$

where $\mathbb{R}[[\ ]]$ denotes formal power series. Let $\mathcal{R}^\mu$ be the functions homogeneous of degree μ with respect to the conic structure in T^*M. All the $\mathcal{R}^\mu$ are filtered by the degree of the lowest order polynomial term in the Taylor expansion in q, r, s. We denote this filtration by

$$\mathcal{R}^\mu = \mathcal{R}_0^\mu \supset \mathcal{R}_1^\mu \supset \cdots.$$

We regard P as a symmetric matrix with coefficients in $\mathcal{R}_1^{1/2}$ and write $P \in S^2(\mathcal{R}_1^{1/2})$. Again we will write the proof out in the $++$ case, the other cases are similar. Our equations can be written as the condition that

$$(5.4) \qquad E_+ : \begin{pmatrix} q+r & s \\ s & q-r \end{pmatrix} \to (\{q,r\}, \{q,s\}, \{r,s\} - 1) : S^2(\mathcal{R}_1^{1/2}) \to \mathcal{R}^0 \otimes \mathbb{R}^3$$

maps to zero. Above we saw that for suitable $A \in GL(2, \mathcal{R}^0)$ we have

$$E_+(APA^t) \in \mathcal{R}_1^0 \otimes \mathbb{R}^3.$$

Proposition 9. *Assume that $E_+(P) \in \mathcal{R}_k^0 \otimes \mathbb{R}^3$ for some $k > 0$. Then there is a $\delta A \in gl(2, \mathcal{R}_k^0)$ such that for $A = I + \delta A$ we have*

$$E_+(APA^t) \in \mathcal{R}_{k+1}^0 \mathbb{R}^3.$$

Proof. This proof is computational. Let $\delta A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with $a, b, c, d \in \mathcal{R}_k^0$, $k > 0$. From (5.1) we see:

$$(5.5) \qquad \begin{cases} \tilde{q} = (1 + a + d)\, q + (a - d)\, r + (b + c)\, s \bmod \mathcal{R}_{2k+1} \\ \tilde{r} = (a - d)\, q + (1 + a + d)\, r + (b - c)\, s \bmod \mathcal{R}_{2k+1} \\ \tilde{s} = (b + c)\, q + (c - b)\, r + (1 + a + d)\, s \ \bmod \mathcal{R}_{2k+1}. \end{cases}$$

The next step is to work out the Poisson brackets appearing in (5.4). The Poisson brackets map $\mathcal{R}_k \otimes \mathcal{R}_m$ to $\mathcal{R}_{m+k-2}$, but taking the Poisson brackets with special functions one can do better. In particular,

$$(5.6) \qquad f \mapsto \{q, f\} \qquad f \mapsto \{r, f\} - \frac{\partial f}{\partial s} \qquad f \mapsto \{s, f\} + \frac{\partial f}{\partial r}.$$

all map $\mathcal{R}_m$ to $\mathcal{R}_m$.

Combining the previous two sets of equations we get:

$$(5.7) \qquad \begin{cases} \{\tilde{q}, \tilde{r}\} = \{q, r\} + \{a+d, r\}\, q + \{a-d, r\}\, r + \{b+c, r\}\, s \\ \qquad\qquad - (b+c) \bmod \mathcal{R}_{k+1} \\ \{\tilde{q}, \tilde{s}\} = \{q, s\} + \{a+d, s\}\, q + \{a-d, s\}\, r + \{b+c, s\}\, s \\ \qquad\qquad + (a-d) \bmod \mathcal{R}_{k+1} \\ \{\tilde{r}, \tilde{s}\} - 1 = \{r, s\} - 1 + \{a-d, s\}\, q + \{r, b+c\}\, q + \{b-c, r\}\, r \\ \qquad\qquad + \{b-c, s\}\, s + \{a+d, s\}\, r + \{r, a+d\}\, s \\ \qquad\qquad + 2(a+d) \bmod \mathcal{R}_{k+1}. \end{cases}$$

Modulo $\mathcal{R}_{k+1}$ the functions $\tilde{q}, \tilde{r}, \tilde{s}$ depend only on the image of δA in $\mathcal{R}_k/\mathcal{R}_{k+1}$.

We will now define suitable vector spaces and operators to discuss the solubility of these equations. We use the associated graded vector spaces may view $\mathcal{G}R_k = \mathcal{R}_k/\mathcal{R}_{k+1}$, which is isomorphic to the space of homogeneous polynomials in q, r, s of degree k with coefficients in $C^\infty(\Sigma)$. First we solve the first two of (5.7) for

$$\alpha \equiv a - d \quad \text{and} \quad \beta \equiv b + c$$

as follows. Write the equations as $\Phi_j(\alpha, \beta) = \phi_j$, $j = 1, 2$, where

$$\Phi_1(\alpha, \beta) := \beta - \{\alpha, r\}\, r - \{\beta, r\}\, s = \beta + r\, \partial\alpha/\partial s + s\, \partial\beta/\partial s,$$

$$\Phi_2(\alpha, \beta) := \alpha + \{\alpha, s\}\, r + \{\beta, s\}\, s = \alpha + r\, \partial\alpha/\partial r + s\, \partial\beta/\partial r,$$

and

$$\phi_1 := \{q, r\} + \{a+d, r\}\, q,$$

$$\phi_2 := -(\{q, s\} - \{a+d, s\}\, q).$$

In the description of Φ_1 we have used (5.6).

These operators on $\mathcal{G}R_k$ leave the $l+1$-dimensional subspace $\mathcal{G}R_{k,l}$, spanned by the $q^{k-l} r^u s^v$ with $u + v = l$, invariant. The kernel of

$$\Phi = (\Phi_1, \Phi_2) : (\mathcal{G}R_{k,l})^2 \to (\mathcal{G}R_{k,l})^2$$

is spanned by the elements $(r^{l-\mu} s^\mu,\ -r^{l-\mu+1} s^{\mu-1})$, $\mu = 1, \ldots, l$, hence $\dim \ker \Phi = l$. On the other hand, its image is contained in the kernel of the mapping

$$\Psi : (\mathcal{G}R_{k,l})^2 \to \mathcal{G}R_{k,l-1} : (\gamma, \delta) \mapsto \{\gamma, s\} + \{\delta, r\} = \partial\gamma/\partial r - \partial\delta/\partial s.$$

The codimension of $\ker \Psi$ is equal to l, so we get that the range of Φ is equal to the kernel of Ψ.

On the other hand, using the Jacobi identity and the fact that $\{q, r\}$, $\{q, s\}$ and $\{r, s\} - 1$ all belong to $\mathcal{R}_k$, we get that $\Psi(\phi_1, \phi_2) = 0$. The conclusion is that the equations $\Phi_j(\alpha, \beta) = \phi_j$ can indeed be solved.

Finally we solve the third equation of (5.7) for $\gamma \equiv a + d \in \mathcal{G}R_k$, by substituting the solution for $b + c$ and $a - d$ found above in terms of $a + d$. For $b - c$ we can substitute anything. The relevant operator is now:

$$\Xi : \mathcal{G}R_k \to \mathcal{G}R_k : \gamma \mapsto 2\gamma + \{r, \gamma\}\, s - \{s, \gamma\}\, r = 2\gamma + s\, \partial\gamma/\partial s + r\, \partial g/\partial r.$$

This map is linear over $C^\infty(\Sigma)$ and $\Xi(q^\rho r^\sigma s^\tau) = (2 + \sigma + \tau)\, q^\rho r^\sigma s^\tau$. So Ξ is invertible, which allows for unique solubility. $\square$

To prove Proposition 6 we call on Borel's lemma [4], [17] which supplies a smooth matrix function which has the Taylor expansion prescribed by the lemma above.

6. EXAMPLES FROM MATHEMATICAL PHYSICS

We first take a look at the Maxwell equations and show that they can satisfy Assumption 4, with either sign for the sum of the squares of the eigenvalues of L, cf. Ivrii [11]. The Maxwell system in a dielectric medium is a system for $\mathbb{R}^3$ valued functions E_j on $\mathbb{R}^3 \times \mathbb{R}$, which can be written as:

$$\sum_j \varepsilon_{ij}(x)\, \frac{\partial^2 E_j}{\partial t^2} = (\nabla \times \nabla \times E)_i.$$

where $\varepsilon_{ij}(x)$ is the dielectric tensor, which is dependent of the space variables and which forms a symmetric 3 by 3 matrix with 3 positive eigenvalues. Assume that ε is diagonal, then we can write the symbol matrix as:

$$(x, \xi) \to \begin{pmatrix} \varepsilon_1(x)\xi_0^2 - \xi_2^2 - \xi_3^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \varepsilon_2(x)\xi_0^2 - \xi_1^2 - \xi_3^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \varepsilon_3(x)\xi_0^2 - \xi_1^2 - \xi_2^2 \end{pmatrix}.$$

The determinant of the symbol can be expressed as:

$$p = \xi_0^2(bc - \xi_2^2\,\alpha),$$

with

$$\begin{aligned} b &= \varepsilon_2\,\xi_0^2 - \xi_1^2 - \xi_3^2, \\ c &= \varepsilon_1\,\varepsilon_3\,\xi_0^2 - \varepsilon_1\,\xi_1^2 - \varepsilon_3\,\xi_3^2, \\ \alpha &= (\varepsilon_2 - \varepsilon_1)(\varepsilon_3 - \varepsilon_2)\,\xi_0^2 + \varepsilon_2(b - \xi_2^2) + c. \end{aligned}$$

As is customary in optics, we assume that $\xi_0 \neq 0$, and we also assume that the eigenvalues of the dielectric tensor satisfy that $\varepsilon_1(x) < \varepsilon_2(x) < \varepsilon_3(x)$.

If the gradient $\partial p/\partial\xi$ of p with respect to the ξ-variables is equal to zero, then $p = \frac{1}{2}\sum \xi_j\, \partial p/\partial\xi_j = 0$, $\xi_2 = 0$, $b = 0$, $c = 0$. Conversely, if $\xi_2 = b = c = 0$, then the rank of the symbol matrix is equal to one, as is easily verified. The conclusion is that, away from $\xi_0 = 0$, Σ corresponds to $\xi_2 = b = c = 0$, and that outside Σ we even have $\partial p/\partial\xi \neq 0$. Also note that Σ is parametrized by:

$$\xi_2 = 0, \qquad \xi_1^2 = \xi_0^2\,\frac{\varepsilon_3(\varepsilon_2 - \varepsilon_1)}{\varepsilon_3 - \varepsilon_1}, \qquad \xi_3^2 = \xi_0^2\,\frac{\varepsilon_1(\varepsilon_3 - \varepsilon_2)}{\varepsilon_3 - \varepsilon_1},$$

with $\xi_0 \neq 0$. (Cf. Kline and Kay [16] and Landau and Lifshitz [20].)

Using that

$$\alpha = (\varepsilon_2 - \varepsilon_1)(\varepsilon_3 - \varepsilon_2)\,\xi_2^0 > 0 \quad \text{at } \Sigma,$$

we get that the set U where $\alpha > 0$ is a conic open neighborhood of Σ. In U we can write $p = \xi_0^2(bc - a^2)$, with $a := \xi_2\sqrt{\alpha}$. Also, Σ is determined by the equations $a = b = c = 0$ in U.

Because da, db and dc are linearly independent, we see that the rank of D^2p is equal to three. Substituting $\xi_0 b = q + r$, $\xi_0 c = q - r$, $\xi_0 a = s$, we have (3.2). Using (3.4) and scaling at $\xi_0^2 = 1$, the equation for the eigenvalues λ of DH_p reads

$$\lambda(\lambda^2 - \{b, c\}^2 + 4\{c, a\}\{a, b\}) = 0,$$

or equivalently

$$\lambda(\lambda^2 - \{b, c\}^2 + 4\{c, \xi_2\}\{\xi_2, b\}(\varepsilon_2 - \varepsilon_1)(\varepsilon_3 - \varepsilon_2)) = 0.$$

Because $\{f, \xi_2\} = \partial f/\partial x_2$ and $\{b, c\}$ only involves partial derivatives of the ε_j with respect to x_1 and x_3, we get quite simple expressions if the ε_j only depend on x_2. In this case the sum of the squares of the eigenvalues is equal to a positive multiple of

$$\varepsilon_2'[\varepsilon_3(\varepsilon_3 - \varepsilon_2)\,\varepsilon_1' + \varepsilon_1(\varepsilon_2 - \varepsilon_1)\,\varepsilon_3'],$$

in which the prime denotes differentiation with respect to x_2. Clearly both signs can occur.

Next we turn to the equations for elastodynamic waves. These are again a system for an $\mathbb{R}^3$ valued function on $\mathbb{R}^4$. Let $\rho(x)$ be the density of the material, $c_{pqrs}(x)$ the moduli of elasticity and $u_i(x)$ the displacement vector. The equations of motion are:

$$\rho\,\partial_t u_p - c_{pqrs}\,\partial_q\,\partial_s u_r = 0. \tag{6.1}$$

The elasticity constants satisfy pointwise the symmetries: $c_{pqrs} = c_{qprs} = c_{rspq}$ and the positivity $c_{pqrs}\,a_{pq}\,a_{rs} \geq 0$ for any symmetric matrix a_{pq}.

Here we shall show that both signs are again realized in this system. Moreover, we observe once more that the derivatives of the material properties are responsible for the sign.

It takes considerable care to determine the conical points of the characteristic variety of the system (6.1). It is known that in the projectivized cotangent bundle the number of conical points is always even, must lie between 0 and 16 and can take any value satisfying these constraints, see Holm [10].

In a medium of cubic symmetry there are many additional relations between the c_{pqrs} and one can exploit these to compute the sign of the system at the points of multiplicity. In particular we can call on the results of Burridge [6] who computes the Hessian of the determinant of the symbol at these points. Working in a single fibre and putting $\xi_0 = 1$, he finds a point of multiplicity 2 with coordinates $(1, \alpha, \alpha, \alpha)$, for a certain number α. At this point he introduces an orthogonal set of axis, the first with direction $(0, 1, 1, 1)$ and all in the hyperplane $\xi_0 = 1$. In the linear coordinates of this system of axes he gets:

$$p(x, \eta) = R\zeta_1^2 - S(\zeta_2^2 + \zeta_3^2)$$

where R, S are positive functions of the moduli of elasticity which can vary independently. From this we immediately deduce that both signs can occur.

REFERENCES

1. Arnol'd, V.I. – On the interior scattering of waves defined by hyperbolic variational principles. Journal of Geometry and Physics **5** (3), 305–315 (1988).
2. Arnol'd, V.I. – Surfaces defined by hyperbolic equations. Math Notes **44** (112), 489–497 (1988).
3. Arnol'd, V.I. – Singularities of caustics and wave fronts. Kluwer, 1991.
4. Borel, E. – Sur quelques points de la théorie des fonctions. Ann. Sci. École Norm. Sup. **12**, 9–55 (1985).
5. Born, M. and E. Wolf – Principles of Optics. Pergamon Press, 1959.
6. Burridge, R. – The singularity on the plane lids of the wave surface of elastic media with cubic symmetry. Quart. Jour. Mech. and Applied Math. **XX**, part 1, 41–56 (1967).
7. Dencker, N. – On the propagation of polarization in conical refraction. Preprint.
8. Dencker, N. – On the propagation of polarization sets for systems of real principal type. Journal of Functional Analysis **46**, 351–372 (1982).
9. Hamilton, W.R. – Third supplement to an essay on the theory of rays. Trans. Roy. Irish Ac. **17** (1) 1–144 (1937).
10. Holm, P. – Generic elastic media. Physica Scripta **T44**, 122–127 (1992).
11. Ivrii, V.Ja. – On wave fronts of solutions of the system of crystal optics. Soviet Math. Dokl. **18**, 139–141 (1977).
12. Ivrii, V.Ja. – Wave fronts of solutions of symmetric pseudodifferential systems. Siberian Math. J. **20** (1979).
13. Duistermaat, J.J. and L. Hörmander – Fourier integral operators II. Acta. Math. **128**, 183–269 (1972).
14. John, F. – Algebraic conditions for hyperbolicity of systems of partial differential operators. Comm. Pure. Appl. Math. **31**, 89–106 and 787–793 (1978).
15. Khesin, B.A. – Singularities of light hypersurfaces and hyperbolicity sets for systems of pde's. In V.I. Arnol'd, editor, Theory of singularities and its applications. Volume 1 of Adv. Sov. Mathematics, 105–118. AMS Providence, 1990.
16. Kline, M. and I.W. Kay – Electromagnetic Theory and Geometric Optics. Wiley, 1964.
17. Hörmander, L. – The Analysis of Linear Partial Differential Operators III. Springer, 1980.
18. Hörmander, L. – Hyperbolic systems with double characteristics. Comm. Pure Appl. Math. **46**, 261–301 (1993).
19. Lax, P.D. – The multiplicity of eigenvalues. Bull. AMS **6**, 213–215 (1982).
20. Lifschitz, E.M., L.D. Landau and L.P. Pitaevskii – Electrodynamics of Continuous Media. Volume 8 of Course of Theoretical Physics. Pergamon Press, 1984.
21. Melrose, R.B. and G.A. Uhlmann – Microlocal structure of involutive conical refraction. Duke Math. J. **46** (1979).
22. Nuij, W. – A note on hyperbolic polynomials. Math. Scand. **23**, 69–72 (1968).
23. Roels, J. and A. Weinstein – Functions whose poisson brackets are constants. Journal of Meth. Phys. **12** (8), 1482–1486 (1971).